

# Clock analysis

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## Abstract

In tenth grade I wrote a program to determine the times of the day at which the short hand and long hand of the clock have a given angle  $\alpha$  to each other. Lacking math skills, I made the program try out all 1440 minutes of a day and compare the angles at the given times. Today I have the necessary math knowledge, to solve this Problem properly.

## 1 Calculating the time to a given angle

Observe that every minute the long hand of the clock turns by  $\frac{360^\circ}{60} = 6^\circ$ . The short hand turns just by  $\frac{360^\circ}{12 \cdot 60} = 0.5^\circ$  per minute and moves  $\frac{360^\circ}{12} = 30^\circ$  per hour. So if it is  $h$  o'clock and  $m$  minutes, the hands have the angles

$$\begin{aligned}s(h, m) &= 30h + 0.5m \\ l(h, m) &= 6m\end{aligned}$$

The angle between the hands is

$$\begin{aligned}\alpha &= s(h, m) - l(h, m) \\ &= 30h + 0.5m - 6m \\ \Leftrightarrow \alpha - 30h &= (0.5 - 6)m \\ \Leftrightarrow \frac{\alpha - 30h}{-5.5} &= \frac{30h - \alpha}{5.5} = m\end{aligned}$$

This means that to determine the times of the day we simply have to replace  $h$  by the available hours  $\{0, 1, 2, \dots, 11\}$  and calculate the minute  $m$  for this particular hour.

Example: the hands have angle  $\alpha = 90^\circ$  at

$$\begin{aligned}m_0 &= \frac{30 \cdot 0 - 90}{5.5} = -16.\overline{36} \Rightarrow 11 : 43 : 38.\overline{18} \\ m_1 &= \frac{30 \cdot 1 - 90}{5.5} = -10.\overline{90} \Rightarrow 00 : 49 : 05.\overline{45} \\ m_2 &= \frac{30 \cdot 2 - 90}{5.5} = -5.\overline{45} \Rightarrow 01 : 54 : 32.\overline{72} \\ m_3 &= \frac{30 \cdot 3 - 90}{5.5} = 0 \Rightarrow 03 : 00 : 00 \\ &\dots\end{aligned}$$

Consider that for example 2 o'clock  $-5.\overline{45}$  minutes is

$$\begin{aligned}
 & 1 : (60 - 5.\overline{45}) \\
 = & 1 : 54.\overline{54} \\
 = & 1 : 54 + 0.\overline{54} \\
 = & 1 : 54 : 60 \cdot 0.\overline{54} \\
 = & 1 : 54 : 32.\overline{72}
 \end{aligned}$$

## 2 Calculating the repeating time of an angle

Now I was intrigued and wanted to know after what timespan the hands would have the same angle again. At hour  $(h + 1)$  the associated minute would have changed by  $\Delta$  minutes:

$$\begin{aligned}
 \frac{30(h+1) - \alpha}{5.5} &= \Delta + \frac{30h - \alpha}{5.5} \\
 \Leftrightarrow \frac{30(h+1) - \alpha}{5.5} - \frac{30h - \alpha}{5.5} &= \Delta \\
 \Leftrightarrow \frac{30h + 30 - 30h - \alpha + \alpha}{5.5} &= \Delta \\
 \Leftrightarrow \frac{30}{5.5} &= 5.\overline{45} = \Delta
 \end{aligned}$$

$5.\overline{45}$  minutes equal 5 minutes and  $27.\overline{27}$  seconds, so every  $1 : 05 : 27.\overline{27}$  hours the hands have the same angle again - independent of the angle!

Note:  $1 : 05 : 27.\overline{27}$  hours are one eleventh of 12 hours.

## 3 Angle-alteration by $1^\circ$

Looking for more applications of the above results I calculated in which timespan  $c$  the angle between the hands changes by exactly  $1^\circ$ .

Let  $m = \frac{30h - \alpha}{5.5}$ .

$$\begin{aligned}
 \frac{30h - \alpha + 1}{5.5} &= m + c \\
 \Leftrightarrow \frac{30h - \alpha + 1}{5.5} &= \frac{30h - \alpha}{5.5} + c \\
 \Leftrightarrow 30h - \alpha + 1 &= 30h - \alpha + 5.5c \\
 \Leftrightarrow 1 &= 5.5c \\
 \Leftrightarrow c &= \frac{1}{5.5} \text{min} = 10.\overline{90} \text{sec.}
 \end{aligned}$$

## 4 Smiling times

Years after finding these results I learned from a TV quiz-show, that in advertisements for clocks, the clocks always show the time  $1 : 50$ , because the clock

"smiles", which increases the sales figures. Obviously at 1 : 50 the angles of the hands aren't symmetrical, because the long hand has  $(360 - 60)^\circ$ , but the short hand only has  $(60 - 10 * 0.5)^\circ = 55^\circ$ , which caused me to think about the question, at which times the angles of the short hand and the long hand are symmetrical (so  $s(h, m) = 360 - l(h, m)$ ). This is the case, iff

$$\begin{aligned}
 6m &= 360 - (30h + 0.5m) \\
 \Leftrightarrow 12m &= 720 - 60h - m \\
 \Leftrightarrow 13m &= 720 - 60h \\
 \Leftrightarrow m &= \frac{720 - 60h}{13}
 \end{aligned}$$

using this result, we can now explicitly calculate the times, at which the hands are symmetrical:

h	m	default representation
0	55.38	0:55:23.0769
1	50.76	1:50:46.1538
2	46.15	2:46:09.2307
3	41.54	3:41:32.3076
4	36.92	4:36:55.3846
5	32.3	5:32:18.4615
6	27.69	6:27:41.5384
7	23.08	7:23:04.6153
8	18.46	8:18:27.6923
9	13.85	9:13:50.7692
10	9.23	10:09:13.8461
11	4.62	11:04:36.9230
12	0	12:00:00

Note: the difference between one of these times, and the next, is always 55 : 23.0769 minutes, which is a thirteenth of 12 hours.